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Thermal linear response of the chaotic maser model

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Abstract. Small amplitude motion in the chaotic maser model with finite temperature is studied in the context of a mean field approximation. The equilibrium state is constructed for both the normal and superradiant phases. The linear response of the model is obtained in these two cases and analytical expressions for the variation of the corresponding frequencies with temperature are derived. Their behaviour both in the integrable and chaotic limit is discussed. Sum rules are shown to be fulfilled and relative percentages turn out to be temperature dependent. Finally we obtain the dynamical stability conditions for the chaotic maser model at finite temperature.

1. Introduction

The maser model was proposed by Dicke [1] in 1954 and since its discovery much has been learnt about the superradiant phase transition in connection to laser physics. Most of the results presently available in such a context were obtained in the so-called rotating wave approximation (RWA), which amounts to neglecting antiresonant terms in the matter–field interaction. Such an approximation provides for excellent results and should be valid for not too intense radiation fields. The main advantage of working with the RWA is that it allows for exact solutions of the dynamical problem [2] as well as of thermodynamical properties [3], which can also be obtained in a mathematically rigorous form [4]. Recently, however, the antiresonant terms have received a lot of attention, especially due to the fact that they are connected with chaotic behaviour [5]. Up to now, studies with the chaotic version of the model have been performed at zero temperature. The aim of the present investigation is not to study chaos but to discuss the finite temperature dependence of small amplitude motion. The linear response of the system is studied and we show in particular how the properties of collective modes and their stability behave as functions of temperature both in the normal and in the superradiant phase. From the experimental point of view, tunable lasers make it possible to study the interaction of a single atom with a single mode of the electromagnetic field in a cavity, at finite temperature [6]. So the dynamics of the one atom–one field interaction contained in the idealized case of a two-level atom interacting with a single quantized mode of the radiation field, as proposed by Jaynes and Cummings [7], can now be studied experimentally. Therefore, the two-level maser model turns out to be not only a toy model with interesting mathematical properties, but in fact describes a situation which exists experimentally. We think that it is of great interest to study the temperature dependence of the atom–field interaction in this

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model, although the main goal of our work is to present techniques which can be applied to other situations as well. In order to have a clear physical insight into the problem, we resort to a thermal mean field approach [8] which allows for simple analytical results. We study in particular the RPA modes (frequencies and eigenvectors) as functions of temperature and an energy-weighted sum rule for thermal states. The redistribution of the strength between the two collective modes is also displayed. Stability conditions for the superradiant and normal phases are very important in the study of chaotic dynamics. Our results are meant to contribute as a first step in this direction by studying the influence of temperature on the stability conditions: when the critical temperature is reached, the RPA frequencies become complex and the RPA frequencies of the normal phase exhibit precisely the opposite behaviour.

The paper is organized as follows. In section 2 we briefly introduce the model; section 3 contains the derivation of the thermal linear response properties and results. Conclusions are given in section 4.

2. The model

The maser model consists of N identical two-level atoms coupled to an electromagnetic field by means of a dipole interaction. The system is enclosed in a cavity of volume V. The atoms are kept at fixed positions and the dimension of V is much smaller than the wavelength of the field so that all atoms see the same field. The Hamiltonian of the model reads

$$H=\varepsilon a^{+}a+\sum_{j=1}^{N}\left[\varepsilon\sigma_j^z+\frac{G}{N^{1/2}}(a\sigma_j^{+}+a^{+}\sigma_j^{-})+\frac{G'}{N^{1/2}}(a^{+}\sigma_j^{+}+a\sigma_j^{-})\right] \quad (2.1)$$

where the index j refers to the jth atom and with

$$\sigma_j^{\pm}=\sigma_j^x\pm\mathrm{i}\sigma_j^y . \quad (2.2)$$

G and G' are coupling constants, a^{+}, a are Bose operators of a harmonic oscillator mode with frequency ε.

3. Thermal linear response

In the present section we use a variational mean field approach in order to calculate the thermal linear response of the chaotic maser model. The techniques are well known and thoroughly discussed in [8, 9].

3.1. Equilibrium states

We start by constructing the equilibrium state. We follow the mean field approximation, which amounts to neglecting correlations, so that, for instance, $\langle J_{+}a\rangle=\langle J_{+}\rangle\langle a\rangle$. The most general form of the finite temperature mean field density matrix for this model is given by

$$D_0=K\exp h_{\mathrm{MF}} \quad (3.1)$$

where

$$h_{\mathrm{MF}}=\alpha_1 J_z+\alpha_2 J_{+}+\alpha_2^{*}J_{-}+\beta_1 a^{+}a+\beta_2 a^{+}+\beta_2^{*}a. \quad (3.2)$$

The approximation amounts to not including in h_{MF} terms of the type J_+a, J_+a^+, J_-a, J_-a^+. In (3.1), K is a normalization constant. The parameters α_1, β_1 in (3.2) are real and α_2, β_2 complex.

It is now convenient to diagonalize D_0. This may be achieved by the unitary operator

$$U=\exp[\mathrm{i}(\eta J_+ + \eta^* J_- + \xi a^+ + \xi^* a)]. \tag{3.3}$$

If the complex numbers η, η^*, ξ, ξ^* are properly chosen, we have

$$D_0 = U^+ D U \tag{3.4}$$

with

$$D = K\exp(\gamma J_z + \gamma' a^+ a). \tag{3.5}$$

(Of course, D is diagonal in a basis of eigenvectors of J_z and of the number operator a^+a.) We minimize the free energy in order to fix the parameters γ, γ', η, η^*, ξ, ξ^*. This procedure specifies the equilibrium density matrix in the mean field approximation. It may be checked that, for the purpose of obtaining static properties of the system, it is enough to consider

$$\eta = \frac{\theta}{2\mathrm{i}} \tag{3.6}$$

and

$$\xi = \xi^*. \tag{3.7}$$

The free energy of the system can now be analytically computed,

$$\begin{aligned}\beta F &= \beta\,\mathrm{Tr}(DUHU^+) + \mathrm{Tr}(D\ln D)\\ &= \beta\left[\varepsilon\frac{N}{2}\cos\theta\tanh\frac{\gamma}{2} + \frac{\varepsilon\,\mathrm{e}^{\gamma'}}{1-\mathrm{e}^{\gamma'}} + \varepsilon\xi^2\right.\\ &\quad\left. + N^{1/2}(G+G')\xi\sin\theta\tanh\frac{\gamma}{2}\right] + \frac{N}{2}\gamma\tanh\frac{\gamma}{2}\\ &\quad + \gamma'\frac{\mathrm{e}^{\gamma'}}{1-\mathrm{e}^{\gamma'}} + \ln(1-\mathrm{e}^{\gamma'}) - N\ln\left(2\cosh\frac{\gamma}{2}\right)\end{aligned} \tag{3.8}$$

with β being the inverse temperature. Minimizing this expression with respect to all the free parameters considered gives us two solutions:

normal phase

$$\begin{aligned}\theta_0 &= p\\ \gamma_0 &= -\beta\varepsilon\\ \gamma_0' &= -\beta\varepsilon\\ \xi_0 &= 0\end{aligned} \tag{3.9}$$

superradiant phase

$$\cos\theta_0 = -\frac{\varepsilon^2}{(G+G')^2}\frac{1}{\tanh\gamma_0/2}$$
$$\gamma_0 = \frac{\beta(G+G')^2}{\varepsilon}\tanh\gamma_0/2 \tag{3.10}$$
$$\gamma_0' = -\beta\varepsilon$$
$$\xi_0 = -\frac{N^{1/2}}{2}\frac{G+G'}{\varepsilon}\sin\theta_0\tanh\gamma_0/2.$$

We see from the above equations that the existence of the superradiant phase depends on the temperature. For $(G+G')^2 < \varepsilon^2$ no phase transition occurs in the system at any temperature. For $(G+G')^2 > \varepsilon^2$ there is a critical temperature T_c given by the relation (for its inverse β_c)

$$\frac{\varepsilon^2}{(G+G')^2} = -\tanh\beta_c\varepsilon/2 \tag{3.11}$$

at which the system changes from one state to the other (the energy slopes are discontinuous). These results have also been obtained by Hioe [10] using a different procedure. They are completely equivalent to those of Hepp and Lieb [4] if we take $G'=0$.

3.2. *Thermal small amplitude motion*

We turn now to the study of the response of the system in its equilibrium state to a small perturbation. If the system is slightly perturbed it will be described by the time-dependent density matrix

$$D(t) = V^+ D(\gamma_0, \gamma_0') V$$
$$= U^+(\theta_0, \xi_0)\, e^{+iS(t)} D(\gamma_0, \gamma_0')\, e^{-iS(t)} U(\theta_0, \xi_0) \tag{3.12}$$

where $U(\theta_0, \xi_0) \equiv U_0$ and $D(\gamma_0, \gamma_0')$ are, respectively, the unitary operator and the diagonal density matrix defined in the previous section. The operator $S(t)$ is an infinitesimal Hermitian operator of the form

$$S(t) = \eta J_+ + \eta^* J_- + \xi a^+ + \xi^* a. \tag{3.13}$$

We use the notation $V = e^{-iS(t)} U(\theta_0, \xi_0)$. If $S=0$, D reduces to the equilibrium density matrix D_0. A small S describes a small deviation from the equilibrium state. The time evolution of a mixed state in the mean field approximation is obtained from the least action principle

$$\delta\int_{t_1}^{t_2} L\, dt = 0 \tag{3.14}$$

with

$$L = i\,\mathrm{Tr}(DV\dot{V}^+) - \mathrm{Tr}(DVHV^+). \tag{3.15}$$

Expanding L in powers of the operator $S(t)$ we obtain for the second-order Lagrangian

$$L^{(2)} = \frac{\mathrm{i}}{2}\,\mathrm{Tr}(D[S,\dot{S}]) - \tfrac{1}{2}\,\mathrm{Tr}(D[S,[U_0 H U_0^+, S]]) \tag{3.16}$$

$$\begin{aligned} &= \frac{\mathrm{i}}{2}(\eta\dot{\eta}^* - \eta^*\dot{\eta})\bar{J}_z + \frac{\mathrm{i}}{2}(\xi^*\dot{\xi} - \dot{\xi}^*\xi) \\ &\quad + 2\bar{J}_z\left(\cos\theta_0 + 2\frac{G_+}{N^{1/2}}\sin\theta_0\xi_0\right)\eta\eta^* \\ &\quad - \bar{J}_z\left(\frac{G_+}{N^{1/2}}\cos\theta_0 - \frac{G_-}{N^{1/2}}\right)(\eta\xi + \eta^*\xi^*) \\ &\quad + \bar{J}_z\left(\frac{G_+}{N^{1/2}}\cos\theta_0 + \frac{G_-}{N^{1/2}}\right)(\eta^*\xi + \eta\xi^*) - \varepsilon\xi\xi^* \end{aligned} \tag{3.17}$$

where

$$\bar{J}_z = \mathrm{Tr}(DJ_z) = \frac{N}{2}\cos\theta_0 \tanh\frac{\gamma_0}{2}$$

and

$$G_\pm = G \pm G'.$$

From (3.17) the equations of motion (which in this case consist of four linear coupled equations) are obtained and the eigenfrequencies and eigenvectors are determined. The equations read

$$\begin{aligned} &(2\bar{J}_z\omega - a)\eta_- + b\xi_+ - c\xi_- = 0 \\ &(-2\bar{J}_z\omega - a)\eta_+ + b\xi_- - c\xi_+ = 0 \\ &(-\omega + \varepsilon)\xi_- + b\eta_+ - c\eta_- = 0 \\ &(\omega + \varepsilon)\xi_+ + b\eta_- - c\eta_+ = 0 \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} a &= 2\bar{J}_z\left(\cos\theta_0 + 2\frac{G_+}{N^{1/2}}\sin\theta_0\xi_0\right) \\ b &= \bar{J}_z\left(\frac{G_+}{N^{1/2}}\cos\theta_0 - \frac{G_-}{N^{1/2}}\right) \\ c &= \bar{J}_z\left(\frac{G_+}{N^{1/2}}\cos\theta_0 + \frac{G_-}{N^{1/2}}\right). \end{aligned} \tag{3.19}$$

We have used the following ansatz for the eigenvectors:

$$\begin{aligned} \eta &= \eta_-\,\mathrm{e}^{-\mathrm{i}\omega t} + \eta_+^*\,\mathrm{e}^{\mathrm{i}\omega t} \\ \xi &= \xi_-\,\mathrm{e}^{-\mathrm{i}\omega t} + \xi_+^*\,\mathrm{e}^{\mathrm{i}\omega t}. \end{aligned} \tag{3.20}$$

The eigenfrequencies can be expressed analytically,

$$\omega^2 = \frac{(2\bar{J}_z D + A) \pm ((2\bar{J}_z D + A)^2 - 8\bar{J}_z(AD - BC))^{1/2}}{4\bar{J}_z} \tag{3.21}$$

with

$$\begin{aligned} A &= \frac{a^2}{2\bar{J}_z} + b^2 - c^2 \\ B &= \frac{a(b-c)}{2\bar{J}_z} - \varepsilon(b+c) \\ C &= \frac{a(b+c)}{2\bar{J}_z} - \varepsilon(b-c) \\ D &= \frac{b^2-c^2}{2\bar{J}_z} + \varepsilon^2. \end{aligned} \tag{3.22}$$

The dependence of the positive frequencies on temperature is displayed in figure 1 both for the normal phase (figure 1(a)) and for the superradiant phase (figure 1(b)). Notice that in the superradiant case one of the frequencies goes to zero as the critical temperature is reached, becoming complex after this point. We see then that the superradiant phase ceases to be stable and the normal phase is stable from there on (it is straightforward to check that while the superradiant phase exists, the corresponding normal phase frequency is complex). We can generalize these comments on the stability of the system. From (3.14) the three general thermal stability conditions

$$2\bar{J}_z D + A < 0$$

and

$$0 < 8\bar{J}_z(AD - BC) < (2\bar{J}_z D + A)^2 \tag{3.23}$$

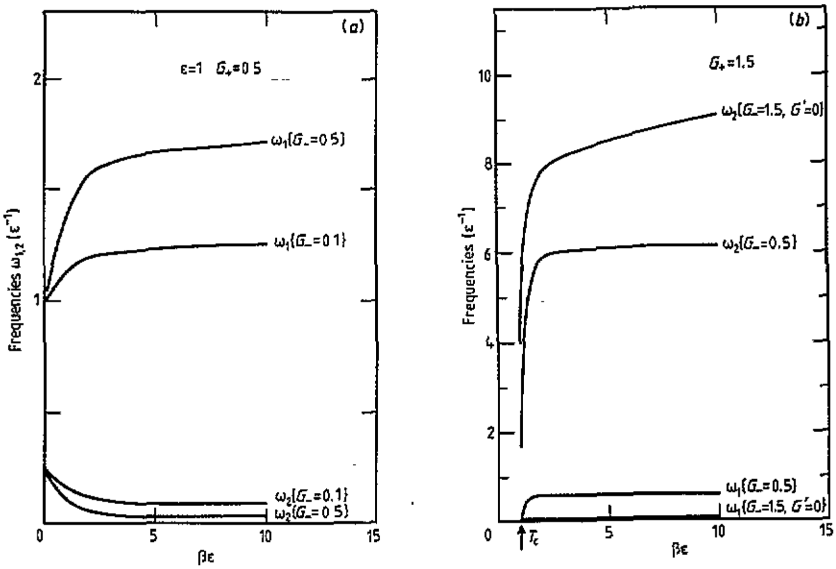

Figure 1. (a) Pairs of eigenfrequencies of the system as functions of the inverse temperature β in the norml phase for two values of the coupling parameter G_-. (b) Same as in (a) but for the superradiant phase. Notice the existence of one constant frequency mode for $G_- = 1.5$, $G' = 0$, which reflects the breaking of the symmetry $[H, J_z + a^+a] = 0$ in the superradiant phase, in the absence of the antiresonant term ($G' = 0$). The inverse of the critical temperature T_c at which the phase transition occurs is shown by an arrow.

are established. When $T \to 0$, these stability conditions reduce to those obtained in [11] from the study of the monodromy matrix of the classical chaotic maser model.

We now present the eigenvectors:

$$\theta_-^i = \frac{-2bc\varepsilon}{(2\bar{J}_z\omega^i - a)(\varepsilon^2 - (\omega^i)^2) + \omega^i(b^2 - c^2) - \varepsilon(b^2 + c^2)}\,\theta_+^i \equiv \alpha\theta_+^i$$

$$\xi_-^i = -\frac{b + c\alpha}{(\varepsilon - \omega^i)}\,\theta_+^i \tag{3.24}$$

$$\xi_+^i = -\frac{b\alpha + c}{(\varepsilon + \omega^i)}\,\theta_+^i.$$

These eigenvectors specify the following boson operators which characterize the collective excited states of the system

$$S_-^i = \theta_-^i J_+ + \theta_+^{i*} J_- + \xi_-^i a^+ + \xi_+^{i*} a$$
$$S_+^i = S_-^{i*}. \tag{3.25}$$

These operators diagonalize the quadratic part of the Hamiltonian [12]. We use the normalization condition to determine θ_+^i.

$$\mathrm{Tr}(D[S_-^i, S_+^j]) = \delta_{ij}. \tag{3.26}$$

Any relevant one-body observable can be expanded in terms of the operators we have just constructed, equations (3.25),

$$\hat{O} = \sum_{i=1}^{2} (c_i S_+^i + c_i^* S_-^i). \tag{3.27}$$

The coefficients c_i are obtained easily as

$$c_i = \mathrm{Tr}(D[S_-^i, \hat{O}]). \tag{3.28}$$

In the same way as the frequency ω is interpreted as the excitation energy of the thermal excited state i with respect to the thermal equilibrium state, the quantity $|c_i|^2$ should be interpreted as the thermal transition probability from the state of thermal equilibrium to the collective state i, induced by the external perturbation $\hat{O}$. The concepts of 'state of thermal equilibrium' and 'thermal collective state' should be understood, of course, in a statistical sense. The corresponding thermal transition strength satisfies the energy weighted sum rule [12]

$$S_1 = \sum_{i=1}^{2} \omega^i |c_i|^2 = \tfrac{1}{2}\mathrm{Tr}(D[\hat{O}, [H, \hat{O}]]). \tag{3.29}$$

The percentage of the sum rule in each one of the two thermal collective modes obtained is clearly a function of temperature also. In figure 2 we show the distribution of the percentages of the sum rule exhausted by the $i = 1$ term, the atomic mode, in the normal phase. The perturbing potentials are assumed to be $\hat{O} = J_x$ (full line) and $\hat{O} = a + a^+$ (dashed line). These situations should correspond to the perturbations of the system by external fluxes of photons and atoms, respectively. For the parameters indicated, only the normal phase exists for all temperatures. On the other hand, by choosing the parameters shown in figure 3, the superradiant phase is the one which is stable below the critical temperature. In a rather small temperature interval, the percentages change abruptly and then stabilize.

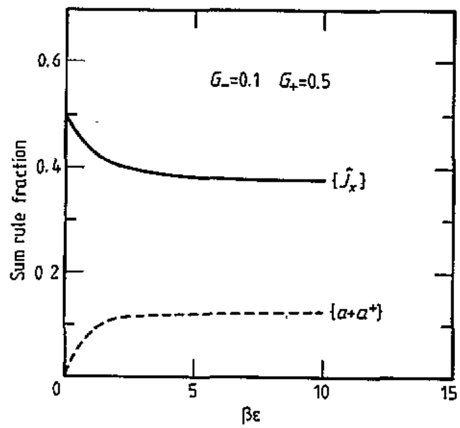

Figure 2. Fractions of the sum rule exhausted by the $i=1$ term, the atomic collective mode, as functions of the inverse temperature β for the coupling parameters $G_+=0.5$, $G_-=0.1$, corresponding to transitions induced by the operator $\hat{O}=J_x$ (full line) and $\hat{O}=a+a^+$ (dashed line). For these values of the parameters only the normal phase exists.

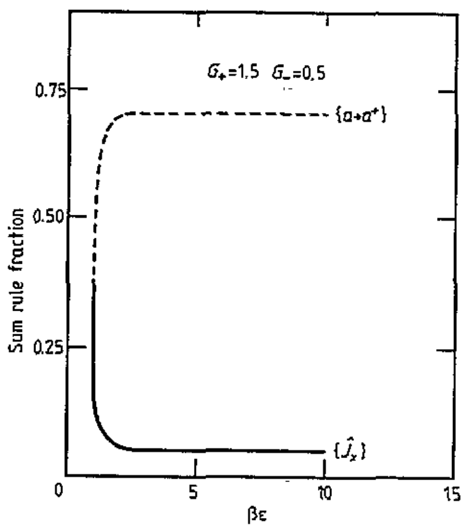

Figure 3. Same as figure 2 but for the superradiant phase with coupling parameters $G_+=1.5$ and $G_-=0.5$.

It may be in order to recall the relevant role which 'sum rules' have played in quantum physics ever since the pioneering work of Thomas, Reiche and Kuhn in atomic physics. These authors were able to derive a useful additive relation between the oscillator strengths associated with the possible excitations of an atom from its ground state $|\phi_0\rangle$ to any of its excited states $|\phi_\nu\rangle$, the transitions being induced by the dipole operator associated with an external plane wave. Similar sum rules hold for any arbitrary quantal system perturbed by an external potential $\hat{O}$. The following relation holds

$$\sum_\nu (E_\nu - E_0)|\langle\phi_\nu|\hat{O}|\phi_0\rangle|^2 = \tfrac{1}{2}\langle\phi_0|[\hat{O},[H,\hat{O}]]|\phi_0\rangle. \tag{3.30}$$

Here, H is the Hamiltonian of the system and E_ν, E_0 are the eigenvalues associated with the excited state $|\phi_\nu\rangle$ and with the ground state $|\phi_0\rangle$. The quantity $(E_\nu - E_0)|\langle\phi_\nu|\hat{O}|\phi_0\rangle|^2$ is the oscillator strength associated with the transition from $|\phi_0\rangle$ to $|\phi_\nu\rangle$. This sum rule must be modified if we wish to describe processes in which the system is not initially in its ground state $|\phi_0\rangle$ but is in a thermal mixed state of equilibrium, described by the density matrix D, such that $D|\phi_\nu\rangle = P_\nu|\phi_\nu\rangle$, $P_\nu \geqslant 0$, $\Sigma_\nu P_\nu = 1$. Of course, P_ν is the ensemble probability of the energy eigenstate $|\phi_\nu\rangle$. Then the following thermal sum rule [13] holds

$$\sum_{\mu,\nu(P_\nu > P_\mu)} (E_\nu - E_\mu)(P_\mu - P_\nu)|\langle\phi_\nu|\hat{O}|\phi_\mu\rangle|^2 = \tfrac{1}{2}\sum_\nu P_\nu\langle\phi_\nu|[\hat{O},[H,\hat{O}]]|\phi_\nu\rangle. \qquad (3.31)$$

The quantity $(E_\nu - E_\mu)(P_\mu - P_\nu)|\langle\phi_\nu|\hat{O}|\phi_\mu\rangle|^2$ is the thermal oscillator strength associated with the transition from $|\phi_\nu\rangle$ to $|\phi_\mu|$. The thermal sum rule (3.29) is precisely the mean field version of the thermal sum rule (3.31).

4. Conclusions

In the present contribution we have analysed the thermal linear response behaviour of the chaotic maser model in the context of a thermal mean field approach. Both responses from normal and superradiant phase have been obtained. The eigenfrequencies can be calculated analytically and are explored as a tool to test the dynamical stability of the system under various conditions. Sum rules are also derived and the relative percentages are shown to be temperature dependent. It would be very interesting to investigate other chaotic models in the same way. The study of the influence of temperature on the behaviour of chaotic systems is in its infancy yet.

Acknowledgments

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References

[1] Dicke R H 1959 *Phys. Rev.* **93** 99
[2] Tavies M and Cummings F W 1968 *Phys. Rev.* **170** 379
[3] Wang Y K and Hioe F T 1973 *Phys. Rev.* A **7** 831
[4] Hepp K and Lieb E H 1973 *Ann. Phys.* **76** 360
[5] Graham R and Hoenerbach M 1986 *Phys. Rev. Lett.* **57** 1378
Lewenkopf C H, Nemes M C, Marvulle V, Pato M P and Wreszinski W S 1991 *Phys. Lett.* **155**A 113
de Aguiar M A M, Furuya, K, Lewenkopf C H and Nemes M C 1991 *Europhys. Lett.* **15** 125
[6] Meschede D, Walther H and Müller G 1985 *Phys. Rev. Lett.* **54** 551
Rempe G, Walther H and Klein N 1987 *Phys. Rev. Lett.* **58** 353
[7] Jaynes E T and Cummings F W 1963 *Proc. IEEE* **51** 89
[8] da Providência J and Fiolhais C 1989 *Nucl. Phys.* A **435** 170
[9] Yamamura M, da Providência J and Kuriyama A 1990 *Nucl. Phys.* A **514** 461
[10] Hioe F T 1973 *Phys. Rev.* A **8** 1440
[11] de Aguiar M A M, Furuya K and Nemes M C 1991 *Quantum Opt.* **3** 305
[12] Ring P and Schuck P 1980 *The Nuclear Many Body Problem* (Berlin: Springer) p 311
[13] Cohen-Tanoudji C, Diu B and Laloë F 1973 *Mécanique Quantique* (Paris: Hermann)